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Refinable and monotone maps revisited

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Abstract

Generalizing results by J. Ford, J. W. Rogers, Jr. and H. Kato we prove that (1) a map f from a G-like continuum onto a graph G is refinable iff f is monotone; (2) a graph G is an arc or a simple closed curve iff every G-like continuum that contains no nonboundary indecomposable subcontinuum admits a monotone map onto G.

We prove that if bonding maps in the inverse sequence of compact spaces are refinable then the projections of the inverse limit onto factor spaces are refinable. We use this fact to show that refinable maps do not preserve completely regular or totally regular continua.

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1. Preliminaries

All spaces in the paper are metric separable and all maps are continuous. A monotone map is a map with connected preimages of connected sets. If $\varepsilon > 0$, then $f : X \to Y$ is an ε*-map* if f is a surjection such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in Y$. We say that X is Y*-like* if for any $\varepsilon > 0$ there exists an ε-map $f : X \to Y$. A map $f : X \to Y$ between compact spaces is *refinable* if for every $\varepsilon > 0$ there is a an ε-map $f_\varepsilon : X \to Y$ such that $d_{\sup}(f, f_\varepsilon) = \sup_{x\in X} d(f(x), f_\varepsilon(x)) < \varepsilon$, where d is a metric in Y (we will say that f and f_ε are ε-close). A *curve* is a 1-dimensional continuum. A *graph* is a space homeomorphic to a 1-dimensional connected polyhedron. An *arc* (a *simple closed curve*) is a space homeomorphic to the unit interval I (the unit circle S^1). A *tree* is a graph containing no simple closed curve. A k*-od* is a tree with exactly one branch point (a *vertex*) and k end-points ($k > 2$). A tree is called an *od* if it is a k-od for some $k > 2$. A continuum X is *regular* (*rational*) if X has a basis of open subsets with finite (countable) boundaries; X is *totally regular* if for every countable subset $D \subset X$ there exists a basis of open subsets of X with finite boundaries omitting D. If the interior of every nondegenerate subcontinuum of a continuum X

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is nonempty, then X is called *completely regular*. A continuum X is *Suslinian* if every collection of nondegenerate mutually disjoint subcontinua of X is countable. The following inclusions are well known (see [7, Proposition 3.2]):

completely regular $\subset$ totally regular $\subset$ regular $\subset$ rational $\subset$ Suslinian.

A continuum is *indecomposable* if it is not the union of two of its proper subcontinua. A continuum is *decomposable* if it is not indecomposable and it is *hereditarily decomposable* if every nondegenerate subcontinuum is decomposable.

Refinable maps were introduced by J. Ford and J.W. Rogers, Jr. in [2]. There is a large literature about refinable maps. The reader is referred to the survey papers [3] and [5].

2. Refinable maps and monotone maps

The following proposition was proved in [2, Corollary 1.2].

Proposition 2.1. *If $f : X \to Y$ is a refinable map between compact spaces and Y is locally connected, then f is monotone.*

Concerning the converse implication, the following is known (the first part was proved in [2, Theorem 5], the second one in [4]).

Proposition 2.2. *A monotone map $f : X \to Y$ is refinable if*

(1) *X is an arc-like continuum and Y is an arc or*
(2) *X is an $S_1 \vee \cdots \vee S_n$-continuum and $Y = S_1 \vee \cdots \vee S_n$, where $S_1 \vee \cdots \vee S_n$ denotes the bouquet of n circles $S_i = S^1$, $i = 1, \ldots, n$, $n \in \mathbb{N}$.*

We are going to extend Proposition 2.2 for all graphs.

Theorem 2.3. *If G is an arbitrary graph and X is a G-like continuum, then a surjection $f : X \to G$ is refinable if and only if f is monotone.*

Proof. Refinable implies monotone by Proposition 2.1.

Assume f is monotone and let $\varepsilon > 0$ be given. Cover G by arcs and ods $A_1, \ldots, A_n$ of diameters $< \frac{\varepsilon}{2}$ such that

(1) if $A_i \cap A_j \neq \emptyset$ for $i \neq j$, then $A_i \cap A_j = \{a_{ij}\}$;
(2) $A_i \cap A_j \cap A_k = \emptyset$ for distinct i, j, k;
(3) if $b \neq b' \in G$ are branch points or end points of G, then there is at least one arc A_i between b and b';
(4) if A_i contains a branch point or an end point, then let a_i denote the point (by condition (3), there can be at most one such point in A_i).

Let $K_i = f^{-1}(A_i)$. We have $K_i \cap K_j \neq \emptyset$ if and only if $A_i \cap A_j \neq \emptyset$. For each $i = 1, \ldots, n$, pick a point $x_i \in K_i$ such that $f(x_i) \in \operatorname{int} A_i$. Let

$$0 < \delta < \min\left\{\frac{\varepsilon}{2}, \min\{d(x_i, K_j)\colon\ i \neq j\}, \min\{d(K_i, K_j)\colon\ K_i \cap K_j = \emptyset\}\right\}$$

and take a δ-map $g : X \to G$. Put $G_i = g(K_i)$. Then $g(x_i) \notin G_j$ for $i \neq j$ and $G_i \cap G_j \neq \emptyset$ if and only if $K_i \cap K_j \neq \emptyset$. If $G_i \cap G_j \neq \emptyset$ for $i \neq j$, then choose a point $g_{ij} \in G_i \cap G_j$. We are going to construct a homeomorphism $h : G \to G$. First put $h(g_{ij}) = a_{ij}$. If G_i is an arc which intersects two different G_j, G_k, then let h homeomorphically map the arc $g_{ij}g_{ik} \subset G_i$ onto the arc $a_{ij}a_{ik} \subset A_i$; if G_i is an arc which intersects only one set G_j, $i \neq j$, then G_i contains an end-point g_i and we define h on the arc $g_{ij}g_i \subset G_i$ to be a homeomorphism onto the arc $a_{ij}a_i \subset A_i$. In the case G_i is not an arc, it is an m-od (by the choice of δ and conditions (1)–(3) which meets m different sets $G_{j_1}, \ldots, G_{j_m}$. Let g_i be the vertex of G_i. We let h homeomorphically map the arcs $g_i g_{ij_k} \subset G_i$ onto the arcs $a_i a_{ij_k} \subset A_i$, respectively. The composition hg is a δ-map, hence an ε-map. To see that it is ε-close to f note that $g(x_i) \in G_i \setminus \bigcup_{j \neq i} G_j$, so both points $f(x_i)$ and $hg(x_i)$ belong to $\operatorname{int} A_i$ whence they are at distance $< \varepsilon$. For $x \neq x_i$, $i = 1, \dots, n$, there is i such that $x \in K_i$. Then $f(x) \in A_i$ and $g(x) \in G_i$. It follows, by the definition of h, that $h(g(x)) \in \bigcup\{A_j\colon A_j \cap A_i \neq \emptyset\}$, thus $d(h(g(x), f(x)) < \varepsilon$. □

The next theorem shows that arcs and simple closed curves play an exceptional role in the classical theory of monotone upper semi-continuous decompositions of curves.

Theorem 2.4. *A graph G is an arc or a simple closed curve if and only if each G-like continuum X which contains no nonboundary indecomposable subcontinuum can be monotony mapped onto G.*

Proof. If G is an arc, then X is arc-like and the Kuratowski's upper semi-continuous monotone decomposition applies to X with the decomposition space an arc (see [6]; alternatively, one can cite the Bing's result [1]). In the case where G is a simple closed curve, X is circle-like and a corresponding upper semi-continuous monotone decomposition of X exists (see [4]).

Assume now that G is neither an arc nor a simple closed curve. We are going to construct a G-like hereditarily decomposable continuum X which admits no monotone map onto G. Choose a free arc $pr \subset G$ (i.e. $\operatorname{int}(pr) = pr \setminus \{p, r\}$) and let $G' = G \setminus \operatorname{int}(pr)$. Denote by G_p and G_r the components of G' which contain p and r, respectively (of course, they can coincide). Define a map $F : \mathbb{R} \to G \times [-1, 1]$, separately on $(-\infty, -1]$, $(-1, 1)$ and $[1, \infty)$, as follows. Since G_p is locally connected, there exists a surjection $f_p : [0, 1] \to G_p$ such that $f_p(0) = f_p(1) = p$. Put $F(x) = (f_p(x - \lfloor x \rfloor), \frac{1}{x})$ for $x \in [1, \infty)$, where $\lfloor x \rfloor$ denotes the integer part of x. Similarly, let $f_r : [0, 1] \to G_r$ such that $f_r(0) = f_r(1) = r$ be a surjection and $F(x) = (f_r(x - \lfloor x \rfloor), \frac{1}{x})$ for $x \in (-\infty, -1]$. Let $t : [-1, 1] \to pr$ be a homeomorphism such that $t(-1) = r$ and $t(1) = p$ and define $F(x) = (t(x), x)$ for $x \in [-1, 1]$. Observe that F is a homeomorphism and $\overline{F(\mathbb{R})} \setminus F(\mathbb{R}) = G' \times \{0\}$. Put $X = \overline{F(\mathbb{R})}$.

For every $\varepsilon > 0$ take $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and define an ε-map $f_\varepsilon : X \to G$ by $f_\varepsilon(g, y) = g$ for $|y| \leqslant \frac{1}{n}$ and $g \in G'$; all other points of X form an arc joining points $(p, \frac{1}{n})$ and $(r, -\frac{1}{n})$, so we let f_ε map this arc homeomorphically onto the arc pr. Thus X is G-like.

To prove that there is no monotone map from X onto G, consider a simple triod in G and note that its preimage by a monotone map would be a generalized triod in X with the nonempty interior in X. Non-boundary subcontinua of X either are contained in the line $F(\mathbb{R})$ or contain one of the sets $G_p \times \{0\}$ or $G_r \times \{0\}$. One can easily check that such continua cannot form a triod in X. □

From Theorems 2.3 and 2.4 we get the following characterization of arcs and simple closed curves among graphs.

Corollary 2.5. *A graph G is an arc or a simple closed curve if and only if for each G-like continuum X which contains no nonboundary indecomposable subcontinuum there is a refinable map from X onto G.*

3. Refinable maps and inverse limits

The aim of this section is to show that if bonding maps of the inverse sequence of compact spaces are refinable, then the projections of the inverse limit are refinable. We provide a proof with necessary details about ε-maps and refinable maps although some of them are standard. The first lemma is obvious.

Lemma 3.1. *If $f : X \to Y$ is an ε-map between compact spaces, then there exists $\delta > 0$ such that* $\operatorname{diam} f^{-1}(U) < \varepsilon$ *for every set $U \subset Y$ of diameter less than δ.*

Lemma 3.2. *If $f_1 : X \to Y$ and $f_2 : Y \to Z$ are refinable maps between compact spaces, then for every $\varepsilon > 0$ there exist ε-maps $g_1 : X \to Y$ and $g_2 : Y \to Z$ such that g_1 is ε-close to f_1, g_2 is ε-close to f_2 and $g_2 g_1$ is an ε-map which is ε-close to $f_2 f_1$.*

Proof. By the uniform continuity of f_2 there exists $0 < \delta_1 < \varepsilon$ such that if $d_Y(y_1, y_2) < \delta_1$, then $d_Z(f_2(y_1), f_2(y_2)) < \frac{\varepsilon}{2}$. Take a δ_1-map g_1 which is δ_1-close to f_1. By Lemma 3.1, there is $0 < \delta_2 < \frac{\varepsilon}{2}$ such that for every set

$U \subset Y$ of diameter less than δ_2 we have $\operatorname{diam} g_1^{-1}(U) < \delta_1$. Let g_2 be a δ_2-map which is δ_2-close to f_2. Then $d_Z(f_2(f_1((x)), g_2(g_1((x))) \leqslant d_Z(f_2(f_1((x)), f_2(g_1((x))) + d_Z(f_2(g_1((x)), g_2(g_1((x))) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ and $\operatorname{diam} g_1^{-1}(g_2^{-1}(z)) < \varepsilon$. □

Corollary 3.3. *If $f_1 : X_1 \to X_2$, $f_2 : X_2 \to X_3, \ldots, f_n : X_n \to X_{n+1}$ are refinable maps between compact spaces, then for every $\varepsilon > 0$ there exist ε-maps $g_1 : X_1 \to X_2$, $g_2 : X_2 \to X_3, \ldots, g_n : X_n \to X_{n+1}$ such that f_i is ε-close to g_i and $g_i \ldots g_1$ is an ε-map which is ε-close to $f_i \ldots f_1$ for $i = 1, \ldots, n$.*

Corollary 3.4. *The composition of finitely many refinable maps between compact spaces is a refinable map.*

Theorem 3.5. *If $X = \varprojlim(X_n, f_n)$ is the inverse limit of compact spaces with refinable bonding maps $f_n : X_{n+1} \to X_n$, then the projection $\pi_n : X \to X_n$ is refinable for each $n \in \mathbb{N}$.*

Proof. Let $\varepsilon > 0$ be given. Consider the metric $\rho((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{d_{X_i}(x_i, y_i)}{2^{i+1}}$ on X, where d_{X_i} is a metric on X_i with $\operatorname{diam} X_i = 1$, for each i. For $n \in \mathbb{N}$, let $N > n$ be such that $\sum_{i=N}^{\infty} \frac{1}{2^{i+1}} < \varepsilon$. The projection π_N is an ε-map. By Lemma 3.1, there exists $0 < \delta < \varepsilon$ such that $\operatorname{diam}_X \pi_N^{-1}(U) < \varepsilon$ for every set $U \subset X_N$ of diameter less than δ. It follows from Corollary 3.3 that there exist δ-maps $g_n, \ldots, g_N$ such that g_i is δ-close to f_i for $i = n, \ldots, N$ and $g_n \ldots g_N$ is a δ-map which is δ-close to $f_n \ldots f_N$. Observe that the map $\pi_n' = g_n \ldots g_N \pi_N$ is ε-close to $\pi_n = f_n \ldots f_N \pi_N$ and $\operatorname{diam} \pi_n'^{-1}(x) = \operatorname{diam} \pi_N^{-1}((g_n, \ldots, g_N)^{-1}(x)) < \varepsilon$. Thus π_n' is an ε-map which is ε-close to π_n. □

4. Completely regular and totally regular continua

Refinable maps preserve regularity of continua because monotone maps do (note that regular continua are locally connected and see Proposition 2.1). Since refinable maps of continua are weakly confluent (i.e., each subcontinuum of the range space is the image of a subcontinuum of the domain) [2, Corollary 1.1], they also preserve Suslinian continua. A long-standing problem (see [2, Question 1]) whether the refinable image of a rational continuum is rational is still open. In this section, using Theorem 3.5, we present two examples showing that neither complete nor total regularity is preserved by refinable maps.

Example 4.1. We construct an inverse limit $X = \varprojlim(X_n, f_n)$ of dendrites X_n as follows. Let X_1 be the universal dendrite for the class of dendrites with points of order $\leqslant 4$ (see Fig. 1).

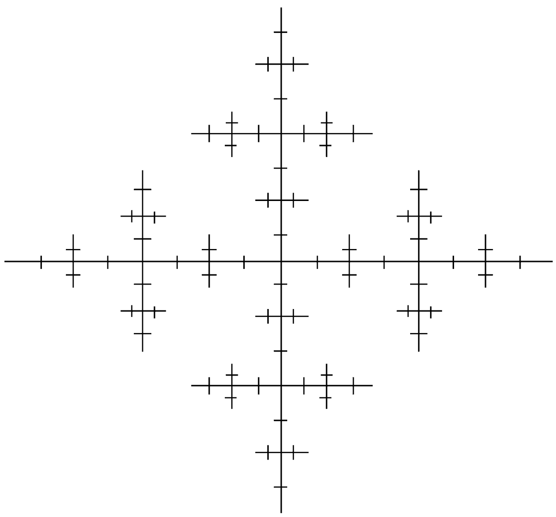

Fig. 1. Dendrite X_1.

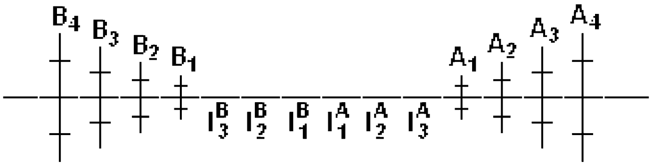

Fig. 2. g_n maps I_n onto close copies of X_1.

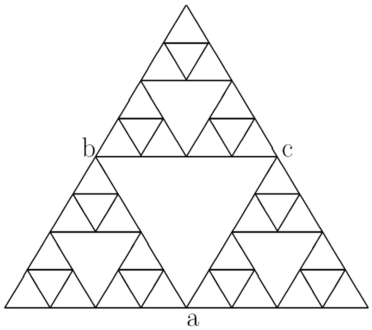

Fig. 3. $X_1 = T$.

It is regular but not completely regular. Let $\{q_1, q_2, \ldots\}$ be a dense in X_1 subset of points of order 2. We create dendrite X_2 by replacing q_1 with a free segment I_1, so that $X_2/I_1 = X_1$ and $f_1 : X_2 \to X_1$ is the quotient map. Then replace the copy of q_2 in X_2 by a free segment I_2 to get a dendrite X_3 such that $X_3/I_2 = X_2$ and let $f_2 : X_3 \to X_2$ be the quotient map, etc. We get $f_n : X_{n+1} \to X_n$ with $f_n(I_n) = \{q_n\}$ and $f_n|_{X_{n+1}\setminus I_n} = \mathrm{id}$ (we identify $X_{n+1} \setminus I_n$ with $X_n \setminus \{q_n\}$) . The dendrite X is completely regular because its every nondegenerate subcontinuum contains a copy of I_n for some n. We will show that all maps f_n are refinable. Given $\varepsilon > 0$, partition I_n into $2k$ congruent consecutive segments $I_1^A, \ldots, I_k^A, I_1^B, \ldots, I_k^B$ of length $< \varepsilon$. In the closure of the component of $X_{n+1} \setminus I_n$ that meets I_k^A choose $k+1$ copies $A_1, \ldots, A_{k+1}$ of X_1 such that $|A_1 \cap I_n| = 1$, $|A_i \cap A_{i+1}| = 1$ for $i = 1, \ldots k$, and $\mathrm{diam}(\bigcup_{i=1}^{k+1} A_i) < \varepsilon$. Similarly in the closure of the other component of $X_{n+1} \setminus I_n$, choose copies $B_1, \ldots, B_{k+1}$ of X_1 such that $|B_1 \cap I_n| = 1$, $|B_i \cap B_{i+1}| = 1$ for $i = 1, \ldots k$ and $\mathrm{diam}(\bigcup_{i=1}^{k+1} B_i) < \varepsilon$ (see Fig. 2). Define $g_n : X_{n+1} \to X_n$ as the identity on $X_{n+1} \setminus (I_n \cup A_1 \cup \ldots \cup A_{k+1} \cup B_1 \cup \ldots \cup B_{k+1})$ and let g_n map (in either way) the set $A_1 \cup \ldots \cup A_{k+1}$ onto $f_n(A_{k+1})$, the set $B_1 \cup \ldots \cup B_{k+1}$ onto $f_n(B_{k+1})$, the interval I_i^A onto $f_n(A_i)$ and I_i^B onto $f_n(B_i)$, for $i = 1, \ldots, k$. Then g_n is an ε-map which is ε-close to f_n. It follows by Theorem 3.5 that π_1 is refinable.

The second example is stronger than the previous one in the sense that it provides a completely regular continuum whose refinable image is not totally regular.

Example 4.2. The idea of a construction of the second example $X = \varprojlim(X_n, f_n)$ is analogous to that of Example 4.1 and is based on the Sierpiński triangular curve T (Fig. 3) which is regular but not totally regular.

Take $X_1 = T$. Continuum X_2 is obtained from X_1 by inserting congruent segments I_a, I_b, I_c in place of three local separating points a, b, c of X_1. We say that points a, b, c appear in the second step of the construction of T. The map $f_1 : X_2 \to X_1$ is the quotient map from X_2 onto $X_1 = X_2/\{I_a, I_b, I_c\}$. Similarly, we get X_{n+1} by replacing new local separating points which appear in the $n+1$-th step of the construction of T by congruent segments. The map $f_n : X_{n+1} \to X_n$ is the quotient map shrinking the segments back to the points (see Fig. 4). Since every nondegenerate subcontinuum of X contains some inserted segment which remains a free arc in X, X is completely regular, so totally regular. To prove that f_n is refinable, we argue as in Example 4.1 using a self-similar structure of T. Each segment $I \subset X_{n+1}$ which f_n shrinks to a point is subdivided into $2k$ congruent subsegments $I_1^A, \ldots, I_k^A, I_1^B, \ldots, I_k^B$ of length $< \varepsilon$. Adjacent to I are two copies T_1 and T_2 of T of diameters $< \frac{\varepsilon}{2}$. Assume T_1 meets I_k^A and T_2 meets I_k^B. Define g_n as the identity on $\overline{X_{n+1} \setminus (I \cup T_1 \cup T_2)}$ (we identify the set with its image under the quotient map f_n) and as a retraction of T_i onto the side of T_i which is disjoint from I, $i = 1, 2$. To define g_n on I, partition T_i into

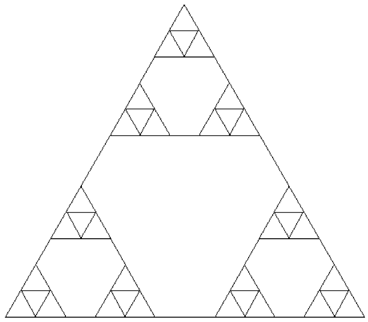

Fig. 4. X_3.

$k-1$ trapezoid-shaped parts $\overline{K(i)_1, \ldots, K(i)_{k-1}}$ such that two consecutive parts have a common side (parallel to a side of T_i) and the subset $R_i = T_i \setminus \bigcup_{j=1}^{k-1} K(i)_j$ similar to T_i and disjoint from I. Then, for $j = 1, \ldots, k-1$, map I_j^A onto $K(1)_j$, I_j^B onto $K(2)_j$, I_k^A onto $R(1)$ and I_k^B onto $R(2)$. The map g_n is an ε-map which is ε-close to f_n.

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